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Synchronization of stochastic mean field networks of Hodgkin-Huxley neurons with noisy channels

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The model

According to the Hodgkin-Huxley model, the dynamics of a neuron's voltage is the result of the passage of ions through its membrane. This ion flux occurs through specific proteins which act as gated channels.

Single Neuron Model

V_t is the voltage of the neuron, $m_t, h_t, n_t \in [0,1]$, are the proportions of open: activation Sodium channels, deactivation Sodium channels and activation Potassium channels respectively.

$$dV_t = F(V_t, m_t, n_t, h_t)dt$$
$$dx_t = [\rho_x(V_t)(1-x_t) - \zeta_x(V_t)x_t]\ dt$$

where, here and in the sequel, x generically represents the m, n, h components and $F : \mathbb{R}\times[0,1]^4 \to \mathbb{R}$, defined by

$$F(V,m,n,h) = I - g_K n^4(V - V_K) - g_{Na} m^3 h(V - V_{Na}) - g_L(V - V_L), \tag{1}$$

represents the effect on the voltage of the ionic channels and of an external current I (assumed constant for simplicity).

Interaction: Synapses

We consider a population of N neurons $\{1,\dots,i,\dots,j,\dots,N\}$, and two type of interactions:

Chemical Synapses: A neurotransmitter is released to the intercellular media from a pre-synaptic neuron to the post-synaptic one through synaptic channels.

With each neuron j we associate a new variable $y^{(j)} \in [0,1]$ which represents its proportion of open synaptic channels at each time.

Chemical synapses coming from neuron j induce on the voltage $V^{(i)}$ of the neuron i an instantaneous variation at time t of

$$-\frac{J_{Ch}}{N} y_t^{(j)}(V_t^{(i)} - V_{rev}),$$

where $J_{Ch} \geq 0$ is the chemical conductance.

Electrical Synapses: Neurons directly connected with each other through an intercellular channel called gap junction.

Pre-synaptic neuron j contributes to the variation of the voltage of post-synaptic neuron i by the amount

$$-\frac{J_E}{N}(V_s^{(i)} - V_s^{(j)}),$$

where $J_E \geq 0$ is the electrical conductance.

Noise

To consider the intrinsic noise present in the ion and neurotransmitter channels we add a noise to the dynamic of the channels:

$$dx_t = \rho_x(V_t)(1-x_t) - \zeta_x(V_t)x_t dt + \sigma_x(V_t, x_t)dW_t^x,$$

where W^x, $x = m, n, h, y$ are independent Brownian motions.

For the specific form of σ_x we refer to Pakdaman et al. [2] and the references therein, where the authors obtain a similar dynamic for the channels as the fluctuations of a Piecewise Deterministic Markov Process.

Model for a Network

We consider the model for a network fully connected studied by Bossy et al. in [1]. The state of the neuron $i = 1,\dots,N$ will represented by $X_t^{(i)} = (V_t^{(i)}, m_t^{(i)}, n_t^{(i)}, h_t^{(i)}, y_t^{(i)})$, and we denote by

$$\bar{V}_t^N = \frac{1}{N}\sum_{i=1}^N V_t^{(i)}, \quad \bar{y}_t^N = \frac{1}{N}\sum_{i=1}^N y_t^{(i)},$$

the empirical means for the voltage and the proportion of open neurotransmitter channels respectively.

The dynamics will be:

$$\begin{aligned} V_t^{(i)} &= V_0^{(i)} + \int_0^t F(V_s^{(i)}, m_s^{(i)}, n_s^{(i)}, h_s^{(i)})ds \\ &\quad - \int_0^t J_E(V_s^{(i)} - \bar{V}_s^N) - J_{Ch}\bar{y}_s^N(V_s^{(i)} - V_{rev})ds, \\ x_t^{(i)} &= x_0^{(i)} + \int_0^t \rho_x(V_s^{(i)})(1 - x_s^{(i)}) - \zeta_x(V_s^{(i)})x_s^{(i)}ds \\ &\quad + \int_0^t \sigma_x(V_s^{(i)}, x_s^{(i)})dW_s^{x,i}, \end{aligned} \tag{2}$$

where $(W^{x,i} : i \in \mathbb{N}, x = m,n,h,y)$ are one dimensional Brownian motions independent of each other and independent of the initial condition.

Simulating the Model

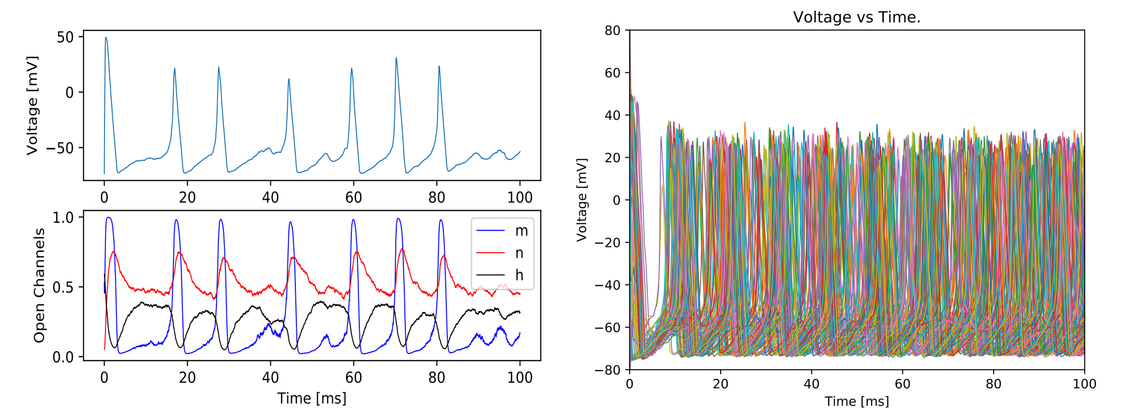

Figure 1: *Left:* The evolution of the Voltage and Channels for one neuron. *Right:* The evolution of the Voltage of 100 neurons J_E small.

Emergence of Synchrony

Numerical simulations show when increasing J_E, the neurons synchronize. This phenomenon is independent of the number of neurons and the phase difference is only affected by the size of the noise.

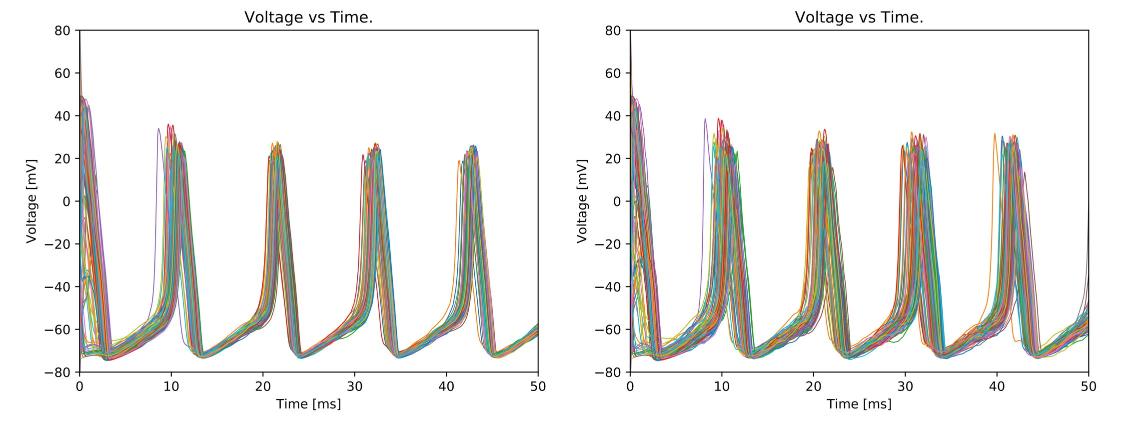

Figure 2: Trajectories for the Voltage of 100 neurons under strong interaction and different level of noise: *Left:* $\sigma = 0.5$. *Right:* $\sigma = 1$.

As a measure of the synchronicity of the network we compute the empirical variance of Voltage of the system (2): $N^{-1}\sum_{i=1}^N (V_t^{(i)} - \bar{V}_t^N)^2$, and to obtain statistically meaningful information, we perform Monte Carlo simulations to estimate the evolution of the expected value of this quantity for different values of N and σ.

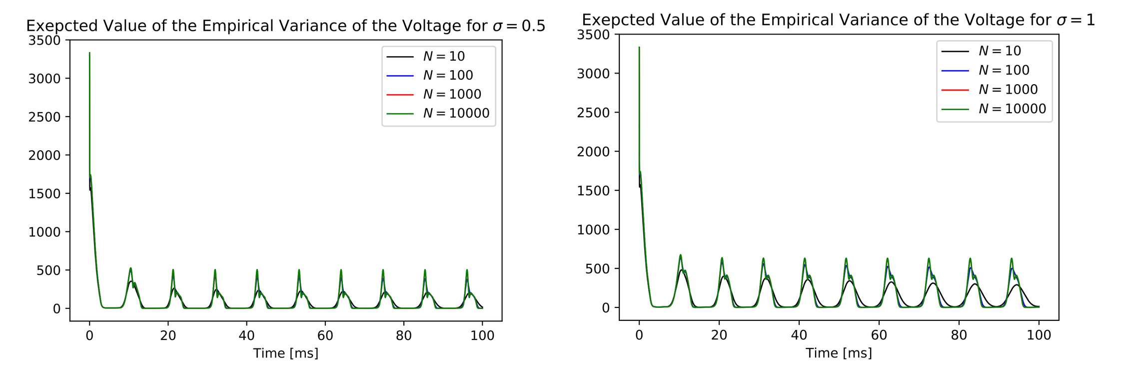

Figure 3: Time evolution of the expected empirical variance of the Voltage for different level of noise: *Left:* $\sigma = 0.5$. *Right:* $\sigma = 1$.

Partial Conclusion: The numerical evidence shows that for big enough J_E the neurons get synchronized, meaning that the empirical variance becomes small in mean. This phenomena seems independent of the size of the network.

Hypothesis (H)

1. ρ_x and ζ_x are strictly positive, locally Lipschitz continuous functions on $\mathbb{R}$. The function $\sigma_x : \mathbb{R}^2 \to \mathbb{R}$ are given by
$$\sigma_x(v,z) = \sigma\sqrt{|\rho_x(v)(1-z) + \zeta_x(v)z|}\chi(z),$$
with $\chi : \mathbb{R} \to [0,1]$ a Lipschitz continuous function with support contained in $[0,1]$ and $\sigma \geq 0$.
2. The *leak conductance* g_L is strictly positive.
3. One has $(m_0^{(i)}, n_0^{(i)}, h_0^{(i)}, y_0^{(i)}) \in [0,1]^4$ a.s.
4. The initial voltages are bounded uniformly in N:
$$\sup_{i=1,\dots,N} |V_0^{(i)}| \leq V_0^{\max} \quad a.s.$$

Reduce Synchronized Dynamics

For $t_1 \geq 0$, let $(\widehat{X}_t^{t_1} : t \geq t_1)$ denote the solution of the ordinary differential equation

$$\begin{aligned} d\widehat{V}_t &= \left[F(\widehat{V}_t, \widehat{m}_t, \widehat{n}_t, \widehat{h}_t) - J_{Ch}\widehat{y}_t(\widehat{V}_t - V_{rev})\right] dt, \\ d\widehat{x}_t &= \left[\rho_x(\widehat{V}_t)(1-\widehat{x}_t) - \zeta_x(\widehat{V}_t)\widehat{x}_t\right] dt \end{aligned} \tag{3}$$

Empirical Case

We denote $(\widehat{X}_t^{t_1,E} : t \geq t_1)$ the solution of (3) with random initial condition $\widehat{X}_{t_1}^{t_1,E} = \bar{X}_{t_1}^N := (\bar{V}_{t_1}^N, \bar{m}_{t_1}^N, \bar{n}_{t_1}^N, \bar{h}_{t_1}^N, \bar{y}_{t_1}^N)$.

Mean Field Case

For $(\mu_t : t \geq 0) \in C([0,\infty), \mathcal{P}_1(\mathbb{R}\times[0,1]^4))$ we denote by $(\widehat{X}_t^{t_1,\infty} : t \geq t_1)$ the solution of (3) with deterministic initial condition $\widehat{X}_{t_1}^{t_1,\infty} = \langle\mu_{t_1}\rangle$, where $\langle\mu_{t_1}\rangle$ is the vector of means of μ_{t_1}.

Main Results

Theorem 1: Synchronization

Assume (H) and that the initial condition is an exchangeable random vector.

a) **Synchronization.** There exist constants $J_E^0 > 0$, $C_{\zeta,\rho}^0 > 0$ and $\lambda^0 > 0$ not depending on $N \geq 1$, $\sigma \geq 0$ or X_0, and there exists a time $t_0 \geq 0$ not depending on $N \geq 1$ or $\sigma \geq 0$, such that for each $J_E > J_E^0$ the solution X of (2) satisfies, for every $t \geq t_0$ and each $i \in \{1,\dots,N\}$:

$$\mathbb{E}\left(|X_t^{(i)} - \bar{X}_t^N|^2\right) \leq \mathbb{E}\left(|X_{t_0}^{(i)} - \bar{X}_{t_0}^N|^2\right)e^{-\lambda^0(t-t_0)} + \sigma^2\frac{C_{\zeta,\rho}^0}{\lambda^0}.$$

In particular, $\limsup_{t\to\infty}\mathbb{E}\left(|X_t^{(i)} - \bar{X}_t^N|^2\right) \leq \sigma^2\frac{C_{\zeta,\rho}^0}{\lambda^0}$.

b) **Synchronized dynamics.** Assume $J_E > J_E^0$. Then, there are constant $K_0, K_0' > 0$ depending only on the parameters of the voltage dynamics in (1) and, for each $\delta \geq 0$, constants $K_\delta, K_\delta' > 0$ depending on the coefficients in (2) and on δ (increasingly) but not on N, such that for every $t_1 \geq t_0$ and each $i \in \{1,\dots,N\}$:

$$\sup_{t_1\leq t\leq t_1+\delta}\mathbb{E}\left(|X_t^{(i)} - \widehat{X}_t^{t_1,E}|^2\right) \leq K_0 \wedge 2\left[\left(K_0'e^{-\lambda_0(t_1-t_0)} + \sigma^2\frac{C_{\zeta,\rho}^0}{\lambda^0}\right)(1+\delta K_\delta) + \delta K_\delta'\frac{\sigma^2}{N}C_{\zeta,\rho}^0\right].$$

Theorem 2: Mean Field Limit

Assume (H) and that for all $N \geq 1$ the initial condition are i.i.d. random vectors with (compactly supported) common law $\mu_0 \in \mathcal{P}(\mathbb{R}\times[0,1]^4)$ not depending on N.

Consider the notations:

$$\begin{aligned} \Phi(w,z,v,u) &:= F(v, u_m, u_n, u_h) - J_E(v-w) - J_{Ch}z(v - V_{rev}) \\ b_x(v,u) &:= \rho_x(v)(1-u_x) - \zeta_x(v)u_x. \\ a_x(v,u) &:= (\rho_x(v)(1-u_x) + \zeta_x(v)u_x)\chi(u_x) \\ \langle\mu\rangle &:= (\langle\mu^V\rangle, (\langle\mu^x\rangle)_{x=m,n,h,y}) \end{aligned}$$

a) There exists $(\mu_t : t \geq 0)$ in $C(\mathbb{R}^+; \mathcal{P}_2(\mathbb{R}\times[0,1]^4))$ a global solution (in the sens of distribution) of the non linear McKean-Vlasov Fokker Planck equation

$$\partial_t\mu_t = \partial_v\left(\Phi(\langle\mu_t^V\rangle, \langle\mu_t^y\rangle, \cdot, \cdot)\mu_t\right) + \sum_{x=m,n,h,y}\frac{1}{2}\sigma^2\partial^2_{u_x u_x}(a_x\mu_t) - \partial_{u_x}(b_x\mu_t)$$

with initial condition μ_0.

b) For each $T > 0$, the process of empirical measures of the system (2) $(\mu_t^N = N^{-1}\sum_{i=1}^N \delta_{X_t^{(i)}} : t \in [0,T])$ converges in law on $C([0,T]; \mathcal{P}_2(\mathbb{R}\times[0,1]^4))$, when N tends to infinity, to a deterministic and uniquely determined flow of probability measures $(\mu_t : t \in [0,T])$ defined on part a).

c) There is a constant $C(T) > 0$ depending on $V_0^{\max}$, $T > 0$ and on the coefficients of system (2), but not on N, such that

$$\sup_{t\in[0,T]}\mathbb{E}\left(\mathcal{W}_2^2(\mu_t^N, \mu_t)\right) \leq C(T)N^{-2/5},$$

where $\mathcal{W}_2$ is the Wasserstein distance of order 2.

Synchronization + Mean Field Limit

Under the assumptions of Theorem 2 and for the same constants as in Theorem 1, whenever $J_E > J_E^0$ we have:

a) For every $t \geq t_0$,

$$\mathcal{W}_2^2(\mu_t, \delta_{\langle\mu_t\rangle}) \leq \mathcal{W}_2^2(\mu_{t_0}, \delta_{\langle\mu_{t_0}\rangle})e^{-\lambda^0(t-t_0)} + \sigma^2\frac{C_{\zeta,\rho}^0}{\lambda^0}.$$

In particular, $\limsup_{t\to\infty}\mathcal{W}_2^2(\mu_t, \delta_{\langle\mu_t\rangle}) \leq \sigma^2\frac{C_{\zeta,\rho}^0}{\lambda^0}$.

b) For every $t_1 \geq t_0$ and $\delta \geq 0$ we have:

$$\sup_{t_1\leq t\leq t_1+\delta}\mathcal{W}_2^2(\mu_t, \delta_{\widehat{X}_t^{t_1,\infty}}) \leq K_0 \wedge 2\left[\left(K_0'e^{-\lambda_0(t_1-t_0)} + \sigma^2\frac{C_{\zeta,\rho}^0}{\lambda^0}\right)(1+\delta K_\delta)\right].$$

References

[1] M. Bossy, O. Faugeras, and D. Talay. Clarification and complement to "mean-field description and propagation of chaos in networks of hodgkin–huxley and fitzhugh–nagumo neurons". *The Journal of Mathematical Neuroscience (JMN)*, 5(1):1–23, 2015.

[2] K. Pakdaman, M. Thieullen, and G. Wainrib. Fluid limit theorems for stochastic hybrid systems with aplication to neuron models. *Adv. in Appl. Probab.*, 42(3):761–794, 09 2010.